HANKEL OPERATORS AND WEAK FACTORIZATION FOR HARDY-ORLICZ SPACES

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This paper is dedicated to the memory of Andrzej Hulanicki who was a colleague, a friend we will never forget.

Abstract. We study the holomorphic Hardy-Orlicz spaces $\mathcal{H}^\Phi(\Omega)$, where Ω is the unit ball or, more generally, a convex domain of finite type or a strictly pseudoconvex domain in $\mathbb{C}^n$. The function Φ is in particular such that $\mathcal{H}^1(\Omega) \subset \mathcal{H}^\Phi(\Omega) \subset \mathcal{H}^p(\Omega)$ for some $p > 0$. We develop for them maximal characterizations, atomic and molecular decompositions. We then prove weak factorization theorems involving the space $BMOA(\Omega)$.

As a consequence, we characterize those Hankel operators which are bounded from $\mathcal{H}^\Phi(\Omega)$ into $\mathcal{H}^1(\Omega)$.

Introduction

The following work has been motivated by a new kind of factorization in the unit disc, obtained in [BIJZ]. Namely, the product of a function in $BMOA$ with a function in the Hardy space $\mathcal{H}^1$ of holomorphic functions lies in some Hardy-Orlicz space defined in terms of the function $\Phi_1(t) := \frac{t}{\log(e+t)}$. Conversely, every holomorphic function of this Hardy-Orlicz space can be written as the product of a function in $BMOA$ and a function in $\mathcal{H}^1$. This exact factorization relies heavily on the classical factorization theorem through Blaschke products and cannot generalize in higher dimension. On the other hand, it has been proven by Coifman, Rochberg and Weiss in the seventies [CRW] that $\mathcal{H}^p$, for $p \leq 1$, admits weak factorization, namely, $F = \sum_j G_j H_j$ with $\sum_j \|G_j\|_q^p \|\|H_j\|_r^p \leq C_{pq}\|F\|_p^p$ when $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$. This has been extended later on by Krantz and Li to strictly pseudo-convex domains [KL2], then by Peloso, Symesak and the authors of the present paper to convex domains of finite type [BPS1], [GP]. We rely on the methods of these two last papers, which are somewhat simpler, to obtain the weak factorization of Hardy-Orlicz spaces under consideration. Note that such a weak factorization for $\mathcal{H}^p$ is typical of the case $p \leq 1$, in opposition to the case of the unit disc.

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A natural application of such factorizations is the characterization of classes of symbols of Hankel operators. We are able to characterize symbols of Hankel operators mapping continuously Hardy-Orlicz spaces into $\mathcal{H}^1$ for a large class of Hardy-Orlicz spaces containing $\mathcal{H}^1$. We do this for all domains for which we have weak factorization. However weak factorization is a stronger property, since the Hardy-Orlicz spaces under consideration are not Banach spaces. We have given in [BGS] a direct proof of the fact that Hankel operators are bounded on $\mathcal{H}^1$ in the unit ball if and only if their symbol is in the space *LMOA*, without involving Hardy-Orlicz spaces, even if the idea of weak factorization indirectly is present in this Note.

Let us mention, in the same direction, the factorization obtained by Cohn and Verbitski in the disc [CV], which allows to characterize those symbols for which the corresponding Hankel operator is bounded from H^2 into some Hardy-Sobolev space. The generalization in higher dimension of their factorization seems much more difficult than ours.

At the end of this paper, we state the same theorems for a class of domains in $\mathbb{C}^n$, which includes the strictly pseudoconvex domains and the convex domains of finite type. We explain rapidly how to modify the proofs.

Let us give some notations and describe more precisely the results. Let $\mathbb{B}^n$ be the unit ball and $\mathbb{S}^n$ be the unit sphere in $\mathbb{C}^n$. Let Φ be a continuous, positive and non-decreasing function on $[0,\infty)$. The Hardy-Orlicz space $\mathcal{H}^\Phi(\mathbb{B}^n)$ is defined as the space of holomorphic functions f so that

$$\sup_{0<r<1}\int_{\mathbb{S}^n}\Phi(|f(rw)|)\,d\sigma(w)<\infty \tag{1}$$

where $d\sigma$ denotes the surface measure on $\mathbb{S}^n$. We recover Hardy spaces $\mathcal{H}^p(\mathbb{B}^n)$ when $\Phi(t)=t^p$. We are especially interested in the case $\Phi_p(t)=\frac{t^p}{\log(e+t)^p}$, $0<p\leq 1$ since the space $\mathcal{H}^{\Phi_p}(\mathbb{B}^n)$ arises naturally in the study of pointwise product of functions in $\mathcal{H}^p(\mathbb{B}^n)$ with functions in $BMOA(\mathbb{B}^n)$ inside the unit ball. Indeed, we prove that the product of an $\mathcal{H}^p(\mathbb{B}^n)$-function and of a $BMOA(\mathbb{B}^n)$-function belongs to the space $\mathcal{H}^{\Phi_p}(\mathbb{B}^n)$ and, conversely, that there is weak factorization.

We will restrict to concave functions Φ, which satisfy an additional assumption so that $\mathcal{H}^1(\mathbb{B}^n)\subset\mathcal{H}^\Phi(\mathbb{B}^n)\subset\mathcal{H}^p(\mathbb{B}^n)$ for some $0<p\leq 1$. In particular, any function f in the Orlicz space $\mathcal{H}^\Phi(\mathbb{B}^n)$ admits a unique boundary function still denoted by f which, by Fatou Theorem, satisfies $\int_{\mathbb{S}^n}\Phi(|f|)d\sigma<\infty$.

We also consider the real Hardy-Orlicz space $H^\Phi(\mathbb{S}^n)$ defined as the space of distributions on $\mathbb{S}^n$ which have an atomic decomposition. More precisely, $H^\Phi(\mathbb{S}^n)$ is the set of distributions f which can be written as $\sum_{j=0}^\infty a_j$, where the a_j's satisfy

adapted cancellation properties, are supported in some ball B_j and are such that $\sum_j \sigma(B_j)\Phi(\|a_j\|_2\sigma(B_j)^{-\frac{1}{2}}) < \infty$.

We first prove usual maximal characterizations of Hardy-Orlicz spaces. As a corollary, we obtain that the Hardy-Orlicz space $\mathcal{H}^\Phi(\mathbb{B}^n)$ continuously embeds into $H^\Phi(\mathbb{S}^n)$, while the Szegö projection is a projection onto $\mathcal{H}^\Phi(\mathbb{B}^n)$. In particular, every $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$ has boundary values that belong to $H^\Phi(\mathbb{S}^n)$, and f may be written in terms of the Szegö projection of its atomic decomposition. The work of Viviani [V] plays a central role: atomic decomposition is proved there for Hardy-Orlicz spaces in the context of spaces of homogeneous type with a restriction on the lower type p of Φ, which, in the case of the unit ball, is the condition $p > \frac{2n}{2n+1}$. We prove the atomic decomposition for all values of p, with the same kind of control of the norm as the one obtained by Viviani.

Since the Szegö projection of an atom is a molecule, we also get a molecular decomposition as in the classical Hardy spaces ([TW] for instance).

The atomic decomposition allows to prove a (weak) factorization theorem on $\mathcal{H}^\Psi(\mathbb{B}^n)$, which coincides with the one for $\mathcal{H}^p(\mathbb{B}^n)$ when $\Psi(t) = t^p$. In particular, we generalize the factorization theorem proved in the disc for $\mathcal{H}^{\Phi_1}$ in [BIJZ]. More precisely, we prove that, given any $f \in \mathcal{H}^\Psi(\mathbb{B}^n)$ there exist $f_j \in \mathcal{H}^\Phi(\mathbb{B}^n), g_j \in BMOA(\mathbb{B}^n)$ such that $f = \sum_{j=0}^\infty f_j g_j$ where Ψ and Φ are linked by the relation $\Psi(t) = \Phi\left(\frac{t}{\log(e+t)}\right)$.

As a consequence, we characterize the class of symbols for which the Hankel operators are bounded from $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $\mathcal{H}^1(\mathbb{B}^n)$. They belong to the dual space of $\mathcal{H}^\Psi(\mathbb{B}^n)$, which can be identified with the *BMOA*-space with weight ρ_Ψ where $\rho_\Psi(t) = \frac{1}{t\Psi^{-1}(1/t)}$. Weighted *BMOA*-spaces have been considered by Janson in the Euclidean space [J]. Here they are defined by

$$BMOA(\rho_\Psi) := \left\{ f \in \mathcal{H}^2(\mathbb{B}^n);\ \sup_B \inf_{P\in\mathcal{P}_N(B)} \frac{1}{\sigma(B)\rho_\Psi(\sigma(B))} \int_B |f-P|^2 d\sigma < \infty \right\}.$$

where the integral is taken on the unit sphere, f stands for the boundary values of the function and balls are defined for the Koranyi metric. Moreover $\mathcal{P}_N(B)$ denotes the set of polynomials of degree $\leq N = N_\Psi$ in an appropriate basis, with N large enough.

When $\Psi = \Phi_1$, this space is usually referred as the space *LMOA* of functions of logarithmic mean oscillation. Duality has been proven in $\mathbb{R}^n$ by [J]. Viviani proves it as a consequence of the atomic decomposition. In the context of holomorphic functions, this is also a consequence of the atomic decomposition and the continuity property of the Szegö projection.

Our method allows us to characterize $BMOA(\rho_\Phi)$ as the class of symbols of Hankel operators which map $\mathcal{H}^\Phi$ into $\mathcal{H}^1_{\text{weak}}$.

As we said, we have chosen to allow the lower type of Φ to be arbitrarily small, and not only of upper type larger than $\frac{2n}{2n+1}$ (for the unit ball of $\mathbb{C}^n$, or for a strictly pseudo-convex domain; for a general convex domain of finite type, the critical index is different). This induces many technical difficulties: for instance, it is not sufficient to deal with atoms with mean 0 and we need extra moment conditions; in parallel, one has to deal with polynomials of positive degree to define the dual space *BMO*, and not only with constants.

Here and in what follows, $\mathcal{H}(\mathbb{B}^n)$ denotes the space of holomorphic functions in $\mathbb{B}^n$. For two functions f and g, we use the notation $f \lesssim g$ when there is some constant c such that $f(w) \leq c\, g(w)$. Here w stands for the parameters that we are interested in (typically, the constant c will only depend on the geometry of the domain under consideration). We use the symbols $\gtrsim$ and $\approx$ analogously.

1. Statements of results

1.1. **Growth functions and Orlicz spaces.** Let us give a precise definition for the *growth functions*, which are used in the definition of Hardy-Orlicz spaces, see also [V].

Definition 1.1. *Let $0 < p \leq 1$. A function Φ is called a growth function of order p if it satisfies the following properties:*

(G1) The function Φ is a homeomorphism of $[0,\infty)$ such that $\Phi(0) = 0$. Moreover, the function $t \mapsto \dfrac{\Phi(t)}{t}$ is non-increasing.

(G2) The function Φ is of lower type p, that is, there exists a constant $c > 0$ such that, for $s > 0$ and $0 < t \leq 1$,

$$\Phi(st) \leq ct^p\Phi(s). \tag{2}$$

We will also say that Φ is a *growth function* whenever it is a growth function of some order $p < 1$. Two growth functions Φ_1 and Φ_2 define the same Hardy-Orlicz spaces when $\Phi_1 \approx \Phi_2$. In particular, the growth function Φ of order p is equivalent to the function $\int_0^t \frac{\Phi(s)}{s} ds$, which is also a growth function of the same order and satisfies the following additional property.

(G3) *The function Φ is concave. In particular, Φ is sub-additive.*

Our typical example $\Phi_p(t) = \frac{t^p}{\log(e+t)^p}$ satisfies (G1) and (G2) for $p \leq 1$. The same is valid for the function $\Phi_{p,\alpha}(t) = t^p(\log(C+t))^{\alpha p}$, provided that C is large enough, for $p < 1$ and any α, or for $p = 1$ and $\alpha < 0$. We modify it as above so that (G3) is also satisfied. In the sequel we will assume that this modification has been done, and use as well the notation Φ_p (or $\Phi_{p,\alpha}$) for the modified function.

Remark 1.2. *When Φ and Ψ are two growth functions, then $\Phi \circ \Psi$ is also a growth function.*

Remark also that Φ is doubling: more precisely,

$$\Phi(2t) \leq 2\Phi(t), \tag{3}$$

a property that will be largely used.

For $(X, d\mu)$ a measure space, we call L^Φ the corresponding Orlicz space, that is, the space of functions f such that

$$\|f\|_{L^\Phi} := \int_X \Phi(|f|)d\mu < \infty.$$

The quantity $\|\cdot\|_{L^\Phi}$ is sub-additive, but is not homogeneous. One may prefer the Luxembourg quasi-norm, which is homogeneous but not sub-additive. It is defined as

$$\|f\|^{\text{lux}}_{L^\Phi} = \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}.$$

It is easily seen that

$$\|f\|^{\text{lux}}_{L^\Phi} \lesssim \min\{\|f\|_{L^\Phi}, \|f\|^p_{L^\Phi}\},$$

while

$$||f||_{L^\Phi} \lesssim \max\{\|f\|^{\text{lux}}_{L^\Phi}, (\|f\|^{\text{lux}}_{L^\Phi})^p\}.$$

Endowed by the distance $||f - g||_{L^\Phi}$, L^Φ is a metric space. When T is a linear continuous operator from L^Φ into the Banach space $\mathcal{B}$, there exists some constant C such that

$$\|Tf\|_{\mathcal{B}} \leq C\|f\|^{\text{lux}}_{L^\Phi} \leq C\|f\|_{L^\Phi}.$$

Conversely, a bounded operator is continuous.

1.2. **Adapted geometry on the unit ball.** Let us recall here the different geometric notions (see [R]) that will be necessary for the description of spaces of holomorphic functions.

For $\zeta \in \mathbb{S}^n$ and $w \in \overline{\mathbb{B}^n}$, we note

$$d(\zeta, w) := |1 - \langle\zeta, w\rangle|.$$

We recall that, when restricted to $\mathbb{S}^n \times \mathbb{S}^n$, this is a quasi-distance. For $\zeta_0 \in \mathbb{S}^n$ and $0 < r < 1$, we note $B(\zeta_0, r)$ the ball on $\mathbb{S}^n$ of center ζ_0 and radius r for the distance d. Recall that $\sigma(B(\zeta_0, r)) \simeq r^n$. In particular,

$$\sigma(B(\zeta_0, \lambda r)) \simeq \lambda^n \sigma(B(\zeta_0, \lambda r)), \tag{4}$$

with constants that do not depend of ζ_0 and r.

For each $\zeta_0 \in \mathbb{S}^n$, we choose an orthonormal basis $v^{(1)}, v^{(2)}, \cdots, v^{(n)}$ in $\mathbb{C}^n$, such that $v^{(1)}$ is the outward normal vector to the unit sphere. In particular, we can choose the canonical basis for the point with coordinates $(1, 0, \cdots, 0)$. Let us call $x_j + iy_j$ the coordinates of $z - \zeta_0$ in this basis. Then $y_1, y_2, \cdots, y_n, x_2, \cdots, x_n$ can be used as coordinates for $\mathbb{S}^n$ in a neighborhood of ζ_0, say in the ball $B(\zeta_0, \delta)$. We

can take δ uniformly for all points ζ_0. We will speak of the *special coordinates related to* ζ_0.

Given $\zeta \in \mathbb{S}^n$ we define the *admissible approach region* $\mathcal{A}_\alpha(\zeta)$ as the subset of $\mathbb{B}^n$ given by

$$\mathcal{A}_\alpha(\zeta) = \{z = rw \in \mathbb{B}^n : \, d(\zeta, w) = |1 - \langle \zeta, w\rangle| < \alpha(1-r)\}.$$

We then define the *admissible maximal function* of the holomorphic function f by $\mathcal{M}_\alpha(f)$

$$\mathcal{M}_\alpha(f)(\zeta) = \sup_{z\in\mathcal{A}_\alpha(\zeta)} |f(z)|. \tag{5}$$

1.3. **Hardy-Orlicz spaces.** Hardy-Orlicz spaces $\mathcal{H}^\Phi(\mathbb{B}^n)$ have been defined in (1). We define on $\mathcal{H}^\Phi(\mathbb{B}^n)$ the (quasi)-norms by

$$\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)} := \sup_{0<r<1} \int_{\mathbb{S}^n} \Phi(|f(rw)|)\, d\sigma(w),$$

$$\|f\|^{\mathrm{lux}}_{\mathcal{H}^\Phi(\mathbb{B}^n)} = \inf\left\{\lambda > 0 : \, \sup_{0<r<1} \int_{\mathbb{S}^n} \Phi\left(\frac{|f(rw)|}{\lambda}\right) d\sigma(w) \le 1\right\},$$

which are finite for $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$ and define the same topology.

The assumptions on the growth function Φ give the inclusions

$$\mathcal{H}^1(\mathbb{B}^n) \subset \mathcal{H}^\Phi(\mathbb{B}^n) \subset \mathcal{H}^p(\mathbb{B}^n). \tag{6}$$

A major property of Hardy spaces is given by the equivalence with definitions in terms of maximal functions, which generalize in our setting.

Theorem 1.3. *Let $\alpha > 0$. There exists a constant $C > 0$ so that, for any $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$,*

$$\|\Phi(\mathcal{M}_\alpha(f))\|_{L^1(\mathbb{S}^n)} \le C\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)} \tag{7}$$

So the two quantities are equivalent.

Next we define the real Hardy-Orlicz spaces on the unit sphere in terms of *atoms*.

For $\zeta_0 \in \mathbb{S}^n$, we consider an orthonormal basis $v^{(1)}, v^{(2)}, \cdots, v^{(n)}$ in $\mathbb{C}^n$ giving rise to special coordinates related to ζ_0. Recall that, if the coordinates for the basis $v^{(j)}$ are denoted by $w_j := x_j + iy_j$, then x_j, for $j \ge 2$ and y_j for $j \ge 1$ define local coordinates of $\mathbb{S}^n$ inside the ball $B(\zeta_0, \delta)$. We call $\mathcal{P}_N(\zeta_0)$ the set of functions on $B(\zeta_0, \delta)$ which are polynomials of degree $\le N$ in these $2n-1$ real coordinates. Remark that $\mathcal{P}_N(\zeta_0)$ does not depend on the choice of $v^{(2)}, \cdots, v^{(n)}$.

Definition 1.4. *A square integrable function a on $\mathbb{S}^n$ is called an atom of order $N \in \mathbb{N}$ associated to the ball $B := B(\zeta_0, r_0)$, for some $\zeta_0 \in \mathbb{S}^n$, if the following properties are satisfied:*

(A1) *supp* aßB;
(A2) $\int_{\mathbb{S}^n} a(\zeta)P(\zeta)d\sigma(\zeta) = 0$ *for every* $P \in \mathcal{P}_N(\zeta_0)$ *when* $r_0 < \delta$.

The second condition is also called *the moment condition*. It is only required for small balls.

We can now define the real Hardy-Orlicz spaces. Recall that the term "real" is related with the fact that the definition makes sense for real functions, and does not require any assumption of holomorphy. Here we consider complex valued functions, since in particular we are interested in the fact that these spaces contain boundary values (in the distribution sense) of holomorphic functions of $\mathcal{H}^\Phi(\mathbb{B}^n)$.

Definition 1.5. *The real Hardy-Orlicz space $H^\Phi(\mathbb{S}^n)$ is the space of distributions f on $\mathbb{S}^n$ which can be written as the limit, in the distribution sense, of series*

$$f = \sum_j a_j, \qquad \sum_j \sigma(B_j)\Phi(\|a_j\|_2 \sigma(B_j)^{-\frac{1}{2}}) < \infty, \tag{8}$$

where the a_j's are atoms of order N, associated to the balls B_j. Here N is an integer chosen so that $N > N_p := 2n(\frac{1}{p} - 1) - 1$.

The (quasi) norm on $H^\Phi(\mathbb{S}^n)$ is defined by

$$\|f\|_{H^\Phi} = \inf\left\{\sum_j \sigma(B_j)\Phi(\|a_j\|_2\sigma(B_j)^{-\frac{1}{2}}) : \, f = \sum_j a_j\right\}. \tag{9}$$

It is also sub-additive. In particular, with the distance between f and g given by $\|f - g\|_{H^\Phi}$, $H^\Phi(\mathbb{S}^n)$ is a complete metric space. It is easily seen that the series in (8) converges in *metric*. Remark that convergence in $H^\Phi(\mathbb{S}^n)$ implies convergence in the sense of distribution.

We will see that the Szegö kernel projects onto the space $\mathcal{H}^\Phi(\mathbb{B}^n)$.

Remark 1.6. *The condition on N guarantees that the Szegö projection of the atom a (or its maximal function) is well defined and has L^Φ norm uniformly bounded in terms of $\Phi(\|a\|_2\sigma(B)^{-\frac{1}{2}})\sigma(B)$. It follows from the theorems below that the space $H^\Phi(\mathbb{S}^n)$ does not depend on $N > N_p$.*

Moreover, we have the following atomic decomposition.

Theorem 1.7. *Let $N \in \mathbb{N}$ be larger than N_p. Given any $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$ there exist atoms a_j of order N such that $\sum_{j=0}^\infty a_j \in H^\Phi(\mathbb{S}^n)$ and*

$$f = P_S\left(\sum_{j=0}^\infty a_j\right) = \sum_{j=0}^\infty P_S(a_j).$$

Moreover

$$\sum_{j=0}^{\infty} \sigma(B_j)\Phi(\|a_j\|_2 \sigma(B_j)^{-\frac{1}{2}}) \approx \|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)}.$$

As in the atomic decomposition of Hardy spaces of $\mathbb{R}^n$, the order of moment conditions of the atoms can be chosen arbitrarily large. Having optimal values has no importance later on, which allows to adapt easily proofs to a class of domains including convex domains of finite type and strictly pseudo-convex domains, for which the optimal values of N_p are different. The fact that atoms may satisfy moment conditions up to an arbitrary large order will play a crucial role for the factorization.

Szegö projections of atoms are best described in terms of molecules, which we introduce now.

Definition 1.8. *A holomorphic function $A \in \mathcal{H}^2(\mathbb{B}^n)$ is called a molecule of order L, associated to the ball $B := B(z_0, r_0) \subset \mathbb{S}^n$, if it satisfies*

$$\|A\|_{\mathrm{mol}(B,L)} := \left(\sup_{r<1} \int_{\mathbb{S}^n} \left(1 + \frac{d(z_0,\xi)^{L+n}}{r_0^{L+n}} \right) |A(r\xi)|^2 \frac{d\sigma(\xi)}{\sigma(B)} \right)^{1/2} < \infty. \tag{10}$$

Proposition 1.9. *For an atom a of order N associated to the ball $B \subset \mathbb{S}^n$, its Szegö projection $P_S(a)$ is a molecule associated to B of any order $L < N+1$. It satisfies*

$$\|A\|_{\mathrm{mol}(B,L)} \lesssim \|a\|_2 \sigma(B)^{-\frac{1}{2}}.$$

Proposition 1.10. *Any molecule A of order L so that $L > L_p := 2n(1/p - 1)$ belongs to $\mathcal{H}^\Phi(\mathbb{B}^n)$ with*

$$\|A\|_{\mathcal{H}^\Phi} \lesssim \Phi(\|A\|_{\mathrm{mol}(B,L)})\sigma(B).$$

The atomic decomposition and the previous propositions have as corollaries the molecular decomposition of functions in $\mathcal{H}^\Phi(\mathbb{B}^n)$, the continuity of the Szegö projection, and the identification of the dual space. Let us begin with molecular decomposition.

Theorem 1.11. *For any $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$, there exists molecules A_j of order L, $L > L_p$, associated to the balls B_j, so that f may be written as*

$$f = \sum_j A_j$$

with $\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)} \approx \sum_j \Phi(\|A_j\|_{\mathrm{mol}(B_j,L)})\sigma(B_j)$.

The continuity of the Szegö projection is also a direct consequence of the atomic decomposition and the fact that an atom is projected into a molecule.

Theorem 1.12. *The Szegö projection extends into a continuous operator,*

$$P_S : H^\Phi(\mathbb{S}^n) \to \mathcal{H}^\Phi(\mathbb{B}^n).$$

Before giving the duality statement, let us first define the generalized $BMO(\varrho)$-spaces as follows. We assume that ϱ is a continuous increasing function from $[0,1]$ onto $[0,1]$, which is of upper type α, that is,

$$\varrho(st) \leq s^\alpha \varrho(t) \tag{11}$$

for $s > 1$, with $st \leq 1$. We then define

$$BMO(\varrho) = \left\{ f \in L^2(\mathbb{S}^n)\ ;\ \sup_B \inf_{P\in\mathcal{P}_N(B)} \frac{1}{\varrho(\sigma(B))\sigma(B)} \int_B |f-P|^2 d\sigma < \infty \right\}.$$

Here, for B a ball of center ζ_B, assumed to be of radius $r < \delta$, we note $\mathcal{P}_N(B) := \mathcal{P}_N(\zeta_B)$. The integer N is taken large enough, say $N > 2n\alpha - 1$. Before going on, let us make some remarks.

Remark 1.13. *Instead of the infimum on $P \in \mathcal{P}_N(B)$, we can take the function $E_N f$, with E_N the orthogonal projection (in $L^2(B)$) onto $\mathcal{P}_N(B)$.*

Remark 1.14. *The definition does not depend on $N > 2n\alpha - 1$. We will not prove this and refer to* [BPS2] *for a proof for $\alpha < 1/2$. It is a consequence of duality and atomic decomposition.*

Remark 1.15. *One may prove that, as in the Euclidean case (see* [J]*) when ϱ is of upper type less than $1/2n$ and satisfies the Dini condition*

$$\int_r^1 \frac{\varrho(s)}{s^2} ds \lesssim \varrho(r),$$

then $BMO(\varrho)$ coincides with the Lipschitz space $\Lambda(\varrho)$, defined as the space of bounded functions such that

$$|f(z) - f(\zeta)| \leq \varrho(d(z,\zeta)^n).$$

Spaces $BMO(\varrho)$ have been introduced by Janson [J] in $\mathbb{R}^n$, and proved to be the dual spaces of maximal Hardy-Orlicz spaces related to the growth function Φ when $\varrho(t) = \varrho_\Phi(t) := \dfrac{1}{t\Phi^{-1}(1/t)}$. With our definition of $H^\Phi(\mathbb{S}^n)$ in terms of atoms, this duality is straightforward, as remarked by Viviani ([V]). For holomorphic Hardy-Orlicz spaces, we have also

Theorem 1.16. *The dual space of $\mathcal{H}^\Phi(\mathbb{B}^n)$ is the space $BMOA(\varrho)$, defined by*

$$BMOA(\varrho) = \left\{ f \in \mathcal{H}^2(\mathbb{B}^n); \sup_B \inf_{P\in\mathcal{P}_N(B)} \frac{1}{\varrho(\sigma(B))\sigma(B)} \int_B |f-P|^2 d\sigma < \infty \right\}$$

where $\varrho(t) = \varrho_\Phi(t) := \dfrac{1}{t\Phi^{-1}(1/t)}$. *The duality is given by the limit as* $r < 1$ *tends to* 1 *of scalar products on spheres of radius* r.

In other terms, $BMOA(\varrho)$ is the space of holomorphic functions of the Hardy space $\mathcal{H}^2(\mathbb{B}^n)$ whose boundary values belong to $BMO(\varrho)$.

1.4. **Products of functions and Hankel operators.** We now have all prerequisites to study the product of a function $h \in \mathcal{H}^\Phi(\mathbb{B}^n)$ with a function in $b \in BMOA(\mathbb{B}^n)$. Remark that, using (6), we already know that the product is well defined as the product of a function of $\mathcal{H}^p(\mathbb{B}^n)$ and a function of $\mathcal{H}^s(\mathbb{B}^n)$ for all $1 < s < \infty$. So it is a function of $\mathcal{H}^q(\mathbb{B}^n)$ for $q < p$. We want to replace this first inclusion by a sharp statement.

Proposition 1.17. *The product maps continuously* $\mathcal{H}^\Phi(\mathbb{B}^n) \times BMOA(\mathbb{B}^n)$ *into* $\mathcal{H}^\Psi(\mathbb{B}^n)$, *where* $\Psi(t) = \Phi\left(\dfrac{t}{\log(e+t)}\right)$.

Proof. We know that Ψ is also a growth function by Remark 1.2. We prove more: using John Nirenberg Inequality, we know that a function b in BMO is also in the exponential class. More precisely, we only use the fact that $b(r\cdot)$ is uniformly in the exponential class, and prove that, for such a function b and for a function $h \in \mathcal{H}^\Phi(\mathbb{B}^n)$, the product $b \times h$ is continuously embedded in $\mathcal{H}^\Psi(\mathbb{B}^n)$. We start from the following elementary inequality, see [BIJZ]. For any $u, v > 0$,

$$\frac{uv}{\log(e+uv)} \leq u + e^v - 1.$$

It follows that

$$\Psi(uv) \lesssim \Phi(u + e^v - 1) \lesssim \Phi(u) + e^v - 1.$$

When u and v are replaced by measurable positive functions on the measure space $(X, d\mu)$, we have, by homogeneity of the Luxembourg norms, the inequality

$$\|fg\|^{\text{lux}}_{L^\Psi} \lesssim \|f\|^{\text{lux}}_{L^\Phi}\|g\|^{\text{lux}}_{\exp L}.$$

We refer to [VT] for more general Hölder inequality on Orlicz spaces.

Let us come back to Hardy spaces. Applying this inequality on each sphere of radius less than 1, we conclude that

$$\|fg\|^{\text{lux}}_{\mathcal{H}^\Psi} \lesssim \|f\|^{\text{lux}}_{\mathcal{H}^\Phi}\|g\|^{\text{lux}}_{\exp L} \lesssim \|f\|^{\text{lux}}_{\mathcal{H}^\Phi}\|g\|_{BMOA}. \tag{12}$$

□

We are going to prove converse statements.

Theorem 1.18. *Let* A *be a molecule associated to the ball* B. *Then* A *may be written as* fg, *where* f *is a molecule and* g *is in* $BMOA(\mathbb{B}^n)$. *Moreover,* f *and* g *may be chosen such that*

$$\|g\|_{BMOA(\mathbb{B}^n)} \lesssim 1, \qquad \|f\|_{\text{mol}(B,L')} \lesssim \frac{\|A\|_{\text{mol}(B,L)}}{\log(e + \sigma(B)^{-1})}$$

when $L' < L$. In particular, if $\Psi(\|A\|_{\mathrm{mol}(B,L)})\sigma(B) \leq 1$, then

$$\Phi(\|f\|_{\mathrm{mol}(B,L')}) \lesssim \Psi(\|A\|_{\mathrm{mol}(B,L)}).$$

Theorem 1.19. *Given any $f \in \mathcal{H}^{\Psi}(\mathbb{B}^n)$ there exist $f_j \in \mathcal{H}^{\Phi}(\mathbb{B}^n)$, $g_j \in BMOA(\mathbb{B}^n)$, $j \in \mathbb{N}$, with the norm of g_j bounded by 1, such that*

$$f = \sum_{j=0}^{\infty} f_j g_j.$$

Moreover, we can take for f_j a molecule and, for $\|f\|_{\mathcal{H}^{\Psi}} \leq 1$, we have the equivalence

$$\|f\|_{\mathcal{H}^{\Psi}} \approx \sum_j \Phi(\|f_j\|_{\mathrm{mol}(B_j,L)})\sigma(B_j).$$

In particular,

$$\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^{\Phi}} \|g_j\|_{BMOA} \lesssim \|f\|_{\mathcal{H}^{\Psi}}.$$

As a corollary, we obtain the following characterization of bounded Hankel operators. Recall that, for $b \in \mathcal{H}^2(\mathbb{B}^n)$, the (small) Hankel operator h_b of symbol b is given, for functions $f \in H^2(\mathbb{B}^n)$, by $h_b(f) = P_S(b\overline{f})$.

Corollary 1.20. *Any Hankel operator h_b extends into a continuous operator from $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ to $\mathcal{H}^1(\mathbb{B}^n)$ if and only if $b \in (\mathcal{H}^{\Psi}(\mathbb{B}^n))' = BMOA(\varrho_{\Psi})$.*

The proof is elementary once we know the previous statements. We give it here.

Proof. Let h_b be a Hankel operator of symbol b. Let us first assume that b belongs to $BMOA(\varrho_{\Psi})$. Then, for any g in $BMOA$, we have

$$\begin{aligned} |\langle h_b(f), g\rangle| &= |\langle P_S(b\overline{f}), g\rangle| = |\langle b, fg\rangle| \\ &\lesssim \|b\|_{BMOA(\varrho_{\Psi})} \|fg\|^{\mathrm{lux}}_{\mathcal{H}^{\Psi}} \lesssim \|b\|_{BMOA(\varrho_{\Psi})} \|f\|^{\mathrm{lux}}_{\mathcal{H}^{\Phi}} \|g\|_{BMOA}. \end{aligned}$$

It follows that h_b is bounded from $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ to $\mathcal{H}^1(\mathbb{B}^n)$, which we wanted to prove.

Conversely, assume now that h_b is bounded from $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ to $\mathcal{H}^1(\mathbb{B}^n)$ and prove that b belongs to the dual of $\mathcal{H}^{\Psi}(\mathbb{B}^n)$. It is sufficient to prove that there exists some constant C such that

$$|\langle b, f\rangle| \leq C$$

when f belongs to a dense subset of functions in $\mathcal{H}^{\Psi}(\mathbb{B}^n)$, with $\|f\|_{\mathcal{H}^{\Psi}} \lesssim 1$. Because of Theorem 1.19, it is sufficient to test on such functions f, which may be written as a finite sum of products $f_j g_j$. More precisely,

$$\begin{aligned} |\langle b, f\rangle| &= |\langle b, \sum f_j g_j| \leq \sum_j |\langle P_S(bf_j), g_j\rangle| \\ &= \sum_j |\langle h_b(f_j), g_j\rangle| \leq |||h_b||| \sum_j \|f_j\|^{\mathrm{lux}}_{\mathcal{H}^{\Phi}} \|g_j\|_{BMOA} \leq C. \end{aligned}$$

It ends the proof. $\square$

All these results may be extended to the more general setting of strictly pseudoconvex domains or of convex domains of finite type in $\mathbb{C}^n$. We give a sketch of the proofs in section 6.

2. Maximal characterizations of Hardy-Orlicz spaces

Let us prove the equivalent characterization of $\mathcal{H}^\Phi$ spaces, given in Theorem 1.3. In order to adapt the proofs given for usual Hardy spaces, we need the following lemma. Here $\mathcal{M}^{HL}$ denotes the Hardy-Littlewood maximal operator related to the distance on the unit sphere. In fact, the statement is valid in the general context of spaces of homogeneous type. In particular we will also use it for the maximal operator on the sphere related to the Euclidean distance.

Lemma 2.1. *Let Φ be a growth function of order p and $\beta < p$. There exists a constant $C > 0$ so that, for any measurable function f,*

$$\int_{\mathbb{S}^n} \Phi\left(\mathcal{M}^{HL}(|f|^\beta)^{\frac{1}{\beta}}\right) d\sigma \leq C \int_{\mathbb{S}^n} \Phi(|f|) d\sigma.$$

Proof. Let us note $g := |f|^\beta$. We only use the fact that

$$t\sigma\left(\mathcal{M}^{HL}(g) \geq t\right) \lesssim \int_{\{g \geq t/2\}} g d\sigma,$$

which is a consequence of the weak (1,1) boundedness of $\mathcal{M}^{HL}$.

Denote by Ψ the function defined by $\Psi(t) := \Phi(t^{\frac{1}{\beta}})$, which is of lower type $p/\beta > 1$. In particular,

$$\int_0^s \frac{\Psi(t)}{t^2} dt = s^{-1} \int_0^1 \frac{\Psi(st)}{t^2} dt \lesssim \frac{\Psi(s)}{s} \tag{13}$$

since $\int_0^1 t^{p/\beta - 2} dt$ is finite. It follows, cutting the integral into intervals $(2^k, 2^{k+1})$, that

$$\sum_{k;\, s > 2^k} 2^{-k} \Psi(2^k) \lesssim \frac{\Psi(s)}{s}. \tag{14}$$

Now, we have to estimate

$$\begin{aligned} \int_{\mathbb{S}^n} \Psi(\mathcal{M}^{HL}(g)) d\sigma &\leq \sum_k \Psi(2^k) \sigma\left(\mathcal{M}^{HL}(g) \geq 2^{k-1}\right) \\ &\lesssim \sum_k 2^{-k} \Psi(2^k) \int_{\{g \geq 2^{k-2}\}} g d\sigma. \end{aligned}$$

Exchanging the integral and the sum and using (14), we obtain that the left hand side is bounded by $C\int_{\mathbb{S}^n}\Psi(g)d\sigma$, which we wanted to prove. $\square$

Proof of Theorem 1.3. We proceed in two steps, as it is classical. Let us note

$$\mathcal{M}_0(f)(\zeta)=\sup_{0<r<1}|f(r\zeta)|$$

the radial maximal function. We first prove that

$$\|\Phi(\mathcal{M}_0(f))\|_{L^1(\mathbb{S}^n)}\leq C\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)}. \tag{15}$$

Let $\beta<p$, Ψ and $g=|f|^\beta$ be as before. The function g is sub-harmonic, and satisfies the condition

$$\sup_{0<r<1}\int_{\mathbb{S}^n}\Psi(g(r\zeta))d\sigma(\zeta)<\infty.$$

We claim that there exists some constant C, independent of g, such that

$$\int_{\mathbb{S}^n}\Psi(\sup_{0<r<1}g(r\zeta))d\sigma(\zeta)\leq C\sup_{0<r<1}\int_{\mathbb{S}^n}\Psi(g(r\zeta)d\sigma(\zeta), \tag{16}$$

which will immediately imply (15). The proof of (16) follows the same lines as in the unit disc. Assume first that g extends into a continuous function on the closed ball and call $\tilde{g}$ the function on the unit sphere that coincides with this extension. With this assumption, the right hand side is the integral of $\Psi(\tilde{g})$. Then it follows from the maximum principle that $g\leq G$, where G is the Poisson integral of $\tilde{g}$. Moreover, we know that $\sup_{0<r<1}g(r\zeta)$ is bounded by the Hardy Littlewood maximal function (for the Euclidean metrics on the unit sphere) of $\tilde{g}$. We conclude for the inequality (16) by using the previous lemma, or its proof, in the context of this maximal function. To conclude for general g, it is sufficient to see that Inequality (15) is valid for g once it is valid for all $g(r\cdot)$, with $0<r<1$.

Let $\tilde{f}$ be the a. e. boundary values of f, which we know to exist since f belongs to $\mathcal{H}^p(\mathbb{B}^n)$ by (6). Remark that once we have done this first step, we also know, using Fatou's lemma, that $\|\Phi(|\tilde{f}|)\|_{L^1(\mathbb{S}^n)}\leq\|f\|_{\mathcal{H}^\Phi}$.

Next, we recall that (see for instance [G], or [St2] for the Euclidean case) that we have the inequality

$$\mathcal{M}_\alpha(f)^\beta\leq C_\alpha\mathcal{M}^{HL}\left(\mathcal{M}_0(f)^\beta\right). \tag{17}$$

We then use Lemma 2.1 to conclude for the proof of Therorem 1.3.

$\square$

We need stronger characterizations of $\mathcal{H}^\Phi(\mathbb{B}^n)$ for the atomic decomposition. First, remark that when looking at the proof of (17), one observes that the constant C_α has a polynomial behavior when α tends to ∞. In the Euclidean

case, details are given in [St2]. This means in particular, using the fact that Φ is doubling, that for some large N_0 and all $\alpha > 0$, we have the inequality

$$\|\Phi(\mathcal{M}_\alpha(f))\|_{L^1(\mathbb{S}^n)} \leq C(1+\alpha)^{N_0}\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)}. \tag{18}$$

Let us consider now the tangential variant of admissible maximal operators, defined by

$$\mathcal{N}_M(f)(\zeta) = \sup_{rw\in\mathbb{B}^n}\left(\frac{1-r}{(1-r)+d(\zeta,w)}\right)^M |f(rw)|. \tag{19}$$

Here $d(\zeta, w)$ denotes the pseudo-distance on $\mathbb{S}^n$, given as before by $d(\zeta, w) := |1 - \langle\zeta, w\rangle|$. We claim that the following identity holds.

$$\|\Phi(\mathcal{N}_M(f))\|_{L^1(\mathbb{S}^n)} \leq C\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)}. \tag{20}$$

Using the definition, we have

$$\begin{aligned}
\mathcal{N}_M f(\zeta) &= \sup_{k\in\mathbb{N}}\sup_{rw\in\mathcal{A}_{2^k}(\zeta)}\left(\frac{1-r}{(1-r)+d(\zeta,w)}\right)^M |f(rw)| \\
&\lesssim \sup_{k\in\mathbb{N}} 2^{-kM}\mathcal{M}_{2^k}f(\zeta).
\end{aligned}$$

It then follows that

$$\|\Phi(\mathcal{N}_M(f))\|_{L^1(\mathbb{S}^n)} \leq \sum_{k\in\mathbb{N}}\|\Phi(2^{-kM}\mathcal{M}_{2^k}f)\|_{L^1(\mathbb{S}^n)} \leq \sum_{k\in\mathbb{N}} 2^{-kMp}\|\Phi(\mathcal{M}_{2^k}f)\|_{L^1(\mathbb{S}^n)}.$$

For $Mp > N_0$ we can conclude after having used (18).

Le us now introduce the *grand maximal function*. Firstly, we define the set of *smooth bump functions at* ζ, which we note $\mathcal{K}_\alpha^N(\zeta)$, as the set of smooth functions φ supported in $B(\zeta_0, r_0)$ for some $\zeta_0 \in \mathcal{A}_\alpha(\zeta)$ and normalized in the following way. In the neighborhood of ζ_0, when we use special coordinates related to ζ_0, the unit sphere coincides with the graph $\Re w_1 = h(\Im w_1, w')$, with $w' = (w_2, \cdots, w_n)$ and h a smooth function. We note $w_j = x_j + y_j$, and consider all derivatives $D^{(k,l)}\varphi$, where $D^{(k,l)}$ consists in k derivatives in x' or y', and l derivatives in y_1. We assume that bump functions $\varphi \in \mathcal{K}_\alpha^N(\zeta)$ satisfy the inequality

$$\sum_{k+l\leq N,}\|D^{(k,l)}\varphi\|_{L^\infty(B(\zeta_0,r_0))}r_0^{k/2+l} \leq \sigma(B)^{-1}.$$

The *grand maximal function* is defined as

$$K_{\alpha,N}(f)(\zeta) = \sup_{\varphi\in\mathcal{K}_\alpha^N(\zeta)}\left|\lim_{r\to1}\int_{S^n} f(r\zeta)\varphi(\zeta)d\sigma(\zeta)\right|. \tag{21}$$

The fact that the limit exists for $f \in \mathcal{H}^\Phi(\mathbb{B}^n) \subset \mathcal{H}^p(\mathbb{B}^n)$ is due to the fact that holomorphic functions in Hardy spaces have boundary values as distributions.

We use the following inequality (see [GP], and [St2] for the Euclidean case).

Lemma 2.2. *With the definitions above, there exist $c = c(\mathbb{B}^n)$ and $\tilde{N} = \tilde{N}(\alpha, N)$ such that*

$$K_{\alpha,N} f(\zeta) \lesssim \mathcal{M}_{c\alpha}(f)(\zeta) + \mathcal{N}_{\tilde{N}}(f)(\zeta).$$

We now turn to the atomic decomposition. We first prove in the next section that holomorphic extensions of functions in $H^\Phi(\mathbb{S}^n)$ are functions of the Hardy-Orlicz space.

3. Atoms and molecules 1.12

We first consider the Szegö projection of atoms and prove the following lemma.

Lemma 3.1. *Let a be an atom of order N associated to the ball $B = B(\zeta_0, r_0)$, and let $A = P_S(a)$. Then A satisfies the following estimates.*

$$\sup_{0<r<1} \int_{B(\zeta_0,2r_0)} \Phi(|A(rw)|) \frac{d\sigma(w)}{\sigma(B)} \lesssim \Phi(\|a\|_2 \sigma(B)^{-\frac{1}{2}}), \tag{22}$$

$$|A(r\zeta)| \lesssim \left(\frac{r_0}{d(\zeta,\zeta_0)}\right)^{n+\frac{N+1}{2}} \|a\|_2 \sigma(B)^{-\frac{1}{2}} \quad \text{for } d(\zeta,\zeta_0) \geq 2r_0. \tag{23}$$

Proof. Let us prove (22). We assumed that Φ is concave. In particular, if $d\mu$ is a probability measure and f a positive function on the measure space $(X, d\mu)$, then we have Jensen Inequality

$$\int_X \Phi(f) d\mu \leq \Phi\left(\int_X f d\mu\right) \leq \Phi\left(\|f\|_{L^2(X,d\mu)}\right). \tag{24}$$

If we use it for the measure $d\sigma$ on $B(z_0, 2r_0)$ after normalization, we find that

$$\sup_{0<r<1} \int_{B(\zeta_0,2r_0)} \Phi(|A(rw)|) \frac{d\sigma(w)}{\sigma(B)} \lesssim \Phi\left(\frac{\|A\|_{\mathcal{H}^2}}{(\sigma(B))^{1/2}}\right). \tag{25}$$

Since the Szegö projection is bounded in L^2, we have the inequality

$$\|A\|_{\mathcal{H}^2} \leq \|a\|_{L^2}$$

and conclude for (22).

The inequality (23) is classical and used for classical Hardy spaces. It is a consequence of the estimates of the Szëgo kernel, which are explicit for the unit ball. Without loss of generality we can assume that $\zeta_0 = (1, 0, \cdots, 0)$, so that the coordinates related to ζ_0 may be taken as the ordinary ones. Otherwise we use the action of the unitary group. In the neighborhood of ζ_0, the unit sphere coincides with the graph $\Re w_1 = h(\Im w_1, w')$, with $w' = (w_2, \cdots, w_n)$. We recall that $S(\zeta, w) = c_n(1 - \zeta.\bar{w})^{-n}$. In the following estimates, we are interested in estimates

on $D_w^{(k,l)} S(r\zeta, (h(t_1, s' + it') + it_1, w'))$, where $D^{(k,l)}$ consists in k derivatives in s' or t', and l derivatives in t_1. It follows from elementary computations that

$$|D_w^{(k,l)} S(r\zeta, (h(t_1, s' + it') + it_1, w'))| \leq C(|\zeta'|^k + |w'|^k)|1 - \zeta.\bar{w}|^{-(n+k+l)}.$$

In particular, for $d(w, \zeta_0) < r_0$ and $\zeta \notin B(\zeta_0, 2r_0)$, we know that $|1 - \zeta.\overline{w}| \simeq |1 - \zeta.\overline{\zeta_0}| \gtrsim r_0$. In particular, we have $|w'| \lesssim |1 - \zeta.\overline{\zeta_0}|^{\frac{1}{2}}$, and the same for $|\zeta'|$. So, the following holds

$$|D_w^{(k,l)} S(r\zeta, (h(t_1, s' + it') + it_1, w'))| \leq C|1 - \zeta.\overline{\zeta_0}|^{-(n+\frac{k}{2}+l)} \tag{26}$$

We use the vanishing moment condition, in the computation of

$$P_S a(r\zeta) = \int S(r\zeta, w)a(w)d\sigma(w),$$

to replace $S(r\zeta, \cdot)$ by $S(r\zeta, \cdot) - P$, where P is its Taylor polynomial at order N. By Taylor's formula, the rest may be bounded by the sum, for $k + l = N + 1$, of the quantities $|t_1|^l |w'|^k |1 - z.\overline{\zeta_0}|^{-(n+\frac{k}{2}+l)}$. Using the fact that $|t_1|^l |w'|^k \lesssim r_0^{\frac{k}{2}+l}$, we have

$$|S(r\zeta, w) - P(w)| \leq C \frac{r_0^{\frac{N+1}{2}}}{d(z, \zeta_0)^{n+\frac{N+1}{2}}}.$$

This gives the result, using the fact that $\sigma(B) \lesssim r_0^n$. $\square$

Proof of Proposition 1.9. The fact that $P_S a$ is a molecule is classical. We give the proof for completeness. Coming back to the definition of $\|P_S(a)\|^2_{\mathrm{mol}(B,L)}$ given in (10), we cut the integral involved into two pieces. We already know that the integral on $B(\zeta_0, 2r_0)$ satisfies the right estimate. So is sufficient to show that

$$\int_{\mathbb{S}^n \setminus B(\zeta_0, 2r_0)} \left(\frac{d(\xi, \zeta_0)}{r_0}\right)^{L+n} \sup_r |P_S a(r\xi)|^2 \frac{d\sigma(\xi)}{\sigma(B)} \lesssim \|a\|_2^2.$$

Using (23), it is a consequence of the estimate

$$\int_{\mathbb{S}^n \setminus B(\zeta_0, 2r_0)} \left(\frac{r_0}{d(\xi, \zeta_0)}\right)^M \frac{d\sigma(\xi)}{\sigma(B)} \leq C \tag{27}$$

for some constant C that does not depend on ζ_0 and r_0, when $M > n$ (see for instance [R]).
$\square$

Proof of Proposition 1.10. Let A be a molecule of order L associated to $B := B(z, r)$. We want to prove that A belongs to $\mathcal{H}^\Phi(\mathbb{B}^n)$ for L large enough, with the estimate

$$\|A\|_{\mathcal{H}^\Phi} \lesssim \Phi(\|A\|_{\mathrm{mol}(B,L)})\sigma(B).$$

Let us note $B_k := B(z, 2^k r)$. It is sufficient to prove that, for g a positive function on the unit sphere,

$$\int_{\mathbb{S}^n} \Phi(g)\frac{d\sigma}{\sigma(B)} \lesssim \Phi\left(\left(\int_{\mathbb{S}^n}\left(\frac{d(z,\xi)}{r}\right)^{L+n} g(\xi)^2\frac{d\sigma(\xi)}{\sigma(B)}\right)^{1/2}\right).$$

Cutting the integral into pieces, it is sufficient to prove that

$$\int_B \Phi(g)\frac{d\sigma}{\sigma(B)} \lesssim \Phi\left(\left(\int_B g(\xi)^2\frac{d\sigma(\xi)}{\sigma(B)}\right)^{1/2}\right),$$

which is a direct consequence of Jensen Inequality (24) as before, and, for $k \geq 1$,

$$\int_{B_k\setminus B_{k-1}} \Phi(g)\frac{d\sigma}{\sigma(B)} \lesssim 2^{-k\varepsilon}\Phi\left(\left(2^{k(L+n)}\int_{B_k\setminus B_{k-1}} g(\xi)^2\frac{d\sigma(\xi)}{\sigma(B)}\right)^{1/2}\right)$$

for some $\varepsilon > 0$. To prove this last inequality, we use again Jensen Inequality (24) for the measure $d\sigma$ on $B_k\setminus B_{k-1}$, divided by its total mass $\sigma(B_k\setminus B_{k-1}) \approx 2^{kn}\sigma(B)$. This gives

$$\int_{B_k\setminus B_{k-1}} \Phi(g)\frac{d\sigma}{\sigma(B)} \lesssim 2^{kn}\Phi\left(\left(2^{-kn}\int_{B_k\setminus B_{k-1}} g(\xi)^2\frac{d\sigma(\xi)}{\sigma(B)}\right)^{1/2}\right).$$

We conclude by using the fact that Φ is of lower type p, which allows to write that $2^{kn}\Phi(t) \lesssim \Phi(2^{kn/p}t)$. It is sufficient to choose $L > n(2/p-2)$. $\square$

4. Proof of the atomic decomposition Theorem 1.7

Let f be a fixed function in $\mathcal{H}^\Phi$. As noticed before, f admits boundary values defined a.e. on $\mathbb{S}^n$, that we still denote by f.

We fix also N an integer larger than N_p.

Let k_0 be the least integer such that

$$\|\Phi(K_{\alpha,M}(f) + \mathcal{M}_\alpha(f))\|_{L^1(\mathbb{S}^n)} \leq 2^{k_0}. \tag{28}$$

For a positive integer k, we define

$$\mathcal{O}_k = \{z \in \mathbb{S}^n : K_{\alpha,M}f(z) + \mathcal{M}_\alpha(f)(z) > 2^{k_0+k}\}. \tag{29}$$

For each k, we then fix a Whitney covering $\{B_i^k\}$ of $\mathcal{O}_k$. As it is usual, one can associate to f an atomic decomposition (see [GL] for a proof for Hardy spaces in the unit ball, we refer also to [GP] for a proof in the general context considered in the last paragraph).

Namely, there exist a function h_0 and atoms b_i^k corresponding to the Whitney covering $\{B_i^k\}$ so that the following equality holds in the distribution sense and almost everywhere.

$$f = h_0 + \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} b_i^k. \tag{30}$$

Here, h_0 is a so called "junk atom" bounded by $c2^{k_0}$ while the b_i^k's are atoms supported in the B_i^k's, bounded by $c2^{k+k_0}$, with moment conditions of order N.

Since $\|b_i^k\|_2\sigma(B_i^k)^{-\frac{1}{2}} \le \|b_i^k\|_\infty$, it is sufficient to prove that

$$\sum_{i,k}\sigma(B_i^k)\Phi(\|b_i^k\|_\infty) < \infty.$$

We have

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sigma(B_i^k)\Phi(\|b_i^k\|_\infty) &\le \sum_{k=0}^{\infty}\Phi(2^{k+k_0})\sigma(\emptyset_k)\\
&\le c\int_1^\infty \frac{\Phi(t)}{t}\sigma\big(\{\zeta\in\mathbb{S}^n : K_\alpha^M f(\zeta)+\mathcal{M}_\alpha(f)(\zeta)\ge t\}\big)dt\\
&\lesssim c\int_1^\infty \Phi'(t)\sigma\big(\{\zeta\in\mathbb{S}^n : K_\alpha^M f(\zeta)+\mathcal{M}_\alpha(f)(\zeta)\ge t\}\big)dt\\
&\le \|\Phi(K_\alpha^M f)\|_{L^1(\mathbb{S}^n)} + \|\Phi(\mathcal{M}_\alpha(f))\|_{L^1(\mathbb{S}^n)}\\
&\le c\|f\|_{\mathcal{H}^\Phi}.
\end{aligned}$$

□

As we pointed out before, the atomic decomposition allows to obtain a lot of result such as the molecular decomposition that we are going to consider now.

5. Factorization Theorem and Hankel operators

Let us prove now the factorization theorem 1.18. Let A be a molecule associated to the ball $B = B(\zeta_0, r)$, with $r < 1$. We write $A = fg$, with

$$g(z) := \log\left(\frac{4}{1-\langle z,\zeta\rangle}\right),$$

where $\zeta := (1-r)\zeta_0$. The constant 4 has been chosen in such a way that g, which is holomorphic on $\mathbb{B}^n$, does not vanish. We first remark that we have the required inequality for f, that is,

$$\|f\|_{\mathrm{mol}(B,L')} \lesssim \frac{\|A\|_{\mathrm{mol}(B,L)}}{\log(e+\sigma(B)^{-1})} \tag{31}$$

for $L' < L$. Indeed, this is a direct consequence of the two inequalities

$$|g(z)| \gtrsim \log(4/r) \simeq \log(e + \sigma(B)^{-1}) \quad z \in B(\zeta_0, 2r),$$

$$|g(z)| \gtrsim \log(e + \sigma(B)^{-1}) \left(\frac{r}{d(\zeta_0, z)}\right)^{\varepsilon} \quad z \notin B(\zeta_0, 2r)$$

for $\varepsilon > 0$. We have used the fact that, for $u > 1$ and $v > e$, one has the inequality

$$\log(uv) \leq 2u^{\varepsilon} \log v.$$

We now prove that g belongs uniformly to $BMOA(\mathbb{B}^n)$ or, equivalently, that $(1-|z|^2)|\nabla g|^2 \simeq \frac{(1-|z|^2)}{|1-\langle z,\zeta\rangle|^2}$ is a Carleson measure with uniform bound. Let $B_\rho = B(x_0, \rho)$ be a ball on the boundary of $\mathbb{B}^n$ and $T(B_\rho)$ be the tent over this ball. We have to prove that

$$\int_{T(B_\rho)} \frac{(1-|z|^2)}{|1-\langle z, \zeta\rangle|^2} dV(z) \lesssim \sigma(B_\rho)$$

with constants that are independent of B_ρ, r and ζ_0, or, which is equivalent,

$$\int_0^\rho \int_{B_\rho} \frac{t}{(d(w,\zeta_0)+t)^2}\, dt\, d\sigma(w) \lesssim \sigma(B_\rho).$$

If $d(x_0, \zeta_0) \geq 2\rho$ then, for $w \in B_\rho$, we have $d(w, \zeta_0) \geq \rho$ and the denominator is bounded below by ρ, which allows to conclude. When $d(x_0, \zeta_0) \leq 2\rho$, then B_ρ is included in $\widetilde{B_\rho} := B(\zeta_0, 3\rho)$ which has a measure comparable to B_ρ. Integrating first in t, we have to prove that

$$\int_{\widetilde{B_\rho}} \log\left(\frac{\rho}{d(\zeta_0, w)}\right) d\sigma(w) \lesssim \sigma(\widetilde{B_\rho}).$$

To prove this last inequality, we cut the ball $\widetilde{B_\rho}$ in dyadic shells. We conclude by using the inequality

$$\sum_{j>0} j\sigma(B(\zeta_0, 2^{-j}\rho)) \lesssim \sigma(B_\rho),$$

which is a consequence of the fact that

$$\sigma(B(z, 2^{-j}\rho)) \lesssim 2^{-jn}\sigma(B(z,\rho)).$$

We have recalled this classical inequality in (4).

Assume now that $\Psi(\|A\|_{\mathrm{mol}(B,L)})\sigma(B) \leq 1$. We use the fact that $\log t \simeq \log \Psi(t)$ to get

$$\|A\|_{\mathrm{mol}(B,L)} \lesssim \log(e + \sigma(B)^{-1})$$

and (31) to conclude that

$$\Phi(\|f\|_{\mathrm{mol}(B,L')}) \lesssim \Psi(\|A\|_{\mathrm{mol}(B,L)}).$$

The weak factorization, that is, Theorem 1.19, follows directly from Theorem 1.11 (molecular decomposition) and Theorem 1.18 (factorization of molecules), with the bound below for the quasi-norm of f in the Hardy-Orlicz space. The bound above uses the direct inequality for molecules, that is Proposition 1.10, and for products, that is (12).

We will give some complements to the characterization of symbols of bounded Hankel operators. If $\exp\mathcal{H}$ denotes the class of holomorphic functions f such that $f(r\cdot)$ is uniformly in the exponential class $\exp L$, then Proposition 1.17 is still valid with $BMOA$ replaced by $\exp\mathcal{H}$. Let us remark that the space $\exp\mathcal{H}$ is the dual space of $P_S(L\log L)$, that is, the space of functions that may be written as $P_S g$, with $g \in L\log L$, equipped with the norm

$$\|h\|_{P_S(L\log L)} := \inf\{\|g\|_{L\log L}\,; h = P_S g\}.$$

Then, looking at the proof of Corollary 1.20, we see that we have as well the following improvement, since $P_S(L\log L)$ is contained in $\mathcal{H}^1$.

Proposition 5.1. *If b belongs to $BMOA(\varrho_\Psi)$, then h_b extends into a continuous operator from $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $P_S(L\log L)$.*

This has been proven by different methods in [BM].

The same reasoning allows to characterize as well the Hankel operators which map $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $\mathcal{H}^1_{\text{weak}}$.

Proposition 5.2. *h_b extends into a continuous operator from $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $\mathcal{H}^1_{\text{weak}}$ if and only if b belongs to $BMOA(\varrho_\Phi)$.*

The necessity of the condition follows from the fact that

$$|\langle b, f\rangle| = |\int_{\mathbb{S}^n} b\bar{f}\,d\sigma| \le \|h_b\|\|f\|_{\mathcal{H}_\Phi},$$

so that b defines a continuous linear form on the space $\mathcal{H}_\Phi$. To prove the sufficiency, it is sufficient to prove that h_b maps $\mathcal{H}^\Phi(\mathbb{B}^n)$ to $P_S(L^1)$ when b is in $BMOA(\varrho_\Phi)$. But the dual of $P_S(L^1)$ identifies with $\mathcal{H}^\infty$. So, using duality, it is sufficient to prove that multiplication by an element of the dual, that is, $\mathcal{H}^\infty$, maps $\mathcal{H}_\Phi$ into itself. This is straightforward.

6. Extension of the results in a general setting

We are now going to give the main points which allow to extend our results in a larger class of domain including strictly pseudoconvex domains and convex domains of finite type. Let Ω be a smooth bounded domain in $\mathbb{C}^n$. Define the Hardy-Orlicz spaces as the space of holomorphic functions f so that

$$\sup_{0<\varepsilon<\varepsilon_0} \int_{\delta(w)=\varepsilon} \Phi(|f|)(w)\,d\sigma_\varepsilon(w) < \infty$$

where Φ is as before of lower type p, $\delta(w)$ is the distance from w to $\partial\Omega$ and $d\sigma_\varepsilon$ the Euclidean measure on the level set $\delta(w) = \varepsilon$. Recall that the usual Hardy space of holomorphic functions $\mathcal{H}^p(\Omega)$ on Ω corresponds to the case $\Phi(t) = t^p$.

6.1. **Geometry of H-domains.**

Definition 6.1. We say that Ω is an H-domain if it is a smoothly bounded pseudoconvex domain of finite type and if, moreover, for each $\zeta \in \partial\Omega$ there exist a neighborhood V_ζ and a biholomorphic map Φ_ζ defined on V_ζ such that $\Phi_\zeta(\Omega \cap V_\zeta)$ is geometrically convex.

We recall that a point $\zeta \in \partial\Omega$ is said to be of finite type if the (normalized) order of contact with $\partial\Omega$ of complex varieties at ζ is finite. By [BS] and our assumption it suffices to consider the order of contact of $\partial\Omega$ at ζ with 1-dimensional complex manifolds, see[BS] and references therein. The domain Ω is said to be of finite type if every point on $\partial\Omega$ is of finite type.We denote by M_Ω the maximum of the types of points on $\partial\Omega$. Notice that the class of H-domains contains both the convex domains of finite type and the strictly pseudoconvex domains.

We describe the geometry of an H-domain Ω.This is done locally, using a partition of unity. Moreover, in a neighborhood of a point $\zeta \in \partial\Omega$, using local coordinates and the assumption, we may in fact assume that Ω is geometrically convex. Thus, we do not lose generality if we assume that it is globally convex. Then, there exist an $\varepsilon_0 > 0$ and a defining function ϱ for Ω such that for $-\varepsilon_0 < \varepsilon < \varepsilon_0$ the sets $\Omega_\varepsilon := \{z \in \mathbb{C}^n : \varrho(z) < \varepsilon\}$ are all convex. Moreover, denote by $U = U_{\varepsilon_0}$ the tubular neighborhood of $\partial\Omega$ given by $\{z \in \mathbb{C}^n : -\varepsilon_0 < \varrho(z) < \varepsilon_0\}$. By taking $\varepsilon_0 > 0$ sufficiently small, we may assume that on $\overline{U}$ the normal projection π of U onto $\partial\Omega$ is uniquely defined. Let $z \in U$ and let v be a unit vector in $\mathbb{C}^n$. We denote by $\tau(z, v, r)$ the distance from z to the surface $\{z' : \varrho(z') = \varrho(z) + r\}$ along the complex line determined by v. One of the basic relations among the quantities defined above is the following.There exists a constant C depending only on the geometry of the domain such that given $z \in U$, any unit vector $v \in \mathbb{C}^n$ that is orthogonal to the level set of the function ϱ and $r \le r_0$ and $\eta < 1$ we have

$$C^{-1}\eta^{1/2}\tau(z, v, r) \le \tau(z, v, \eta r) \le C\eta^{1/M_\Omega}\tau(z, v, r). \tag{32}$$

We next define the *r-extremal orthonormal basis* $\{v^{(1)}, \dots, v^{(n)}\}$ at z, which generalize the choices that we have done for the unit ball. The first vector is given by the direction transversal direction to the level set of ϱ containing z, pointing outward. In the complex directions orthogonal to $v^{(1)}$ we choose $v^{(2)}$ in such a way that $\tau(z, v^{(2)}, r)$ is maximum. We repeat the same procedure to determine the remaining elements of the basis. We set

$$\tau_j(z, r) = \tau(z, v^{(j)}, r).$$

By definition, $\tau_1(z, r) \simeq r$. The polydisc $Q(z, r)$ is now given as

$$Q(z, r) = \{w : |w_k| \le \tau_k(z, r),\ k = 1, \dots, n\}.$$

Here $(w_1, \ldots, w_n)$ are the coordinates determined by r-extremal orthonormal basis $\{v^{(1)}, \ldots, v^{(n)}\}$ at z. Notice that these coordinates $(w_1, \ldots, w_n) = (w_1^{z,r}, \ldots, w_n^{z,r})$ depend on z and on r. They are called *special coordinates at the point z and at scale r*. The quasi-distance is defined by setting

$$d_b(z, w) = \inf\{r : w \in Q(z, r)\}. \tag{33}$$

Notice that by the above properties the sets $Q(z, r)$ are in fact equivalent to the balls in the quasi-distance d_b. We also consider balls on the boundary $\partial\Omega$ defined in terms of d_b. For $\zeta \in \partial\Omega$ and $r > 0$ we set

$$B(\zeta, r) = \{z \in \partial\Omega : d_b(z, \zeta) < r\}.$$

These balls are equivalent to the sets $Q(\zeta, r) \cap \partial\Omega$. Moreover, we define the function d on $\overline{\Omega} \times \overline{\Omega}$ by setting

$$d(z, w) = \delta(z) + \delta(w) + d_b\big(\pi(z), \pi(w)\big)\,, \tag{34}$$

where π is the normal projection of a point z onto the boundary. We now set

$$\tau(z, r) = (\tau_1(z, r) \cdots, \tau_n(z, r)).$$

Then, for α a multiindex, we note

$$\tau^\alpha(z, r) = \prod_{j=1}^{n} \tau_j^{\alpha_j}(z, r).$$

When Ω is strictly pseudoconvex, we have simply $\tau^\alpha(z, r) \simeq r^{\frac{|\alpha|+\alpha_1}{2}}$. Let σ denotes the surface measure on $\partial\Omega$. Then, one has

$$\sigma\big(B(w, r)\big) \simeq \tau^{(1,2,\cdots,2)}(w, r).$$

Moreover, the property (4) is replaced by the double inequality

$$\lambda^n \sigma(\zeta_0, r) \lesssim \sigma(B(\zeta_0, \lambda r)) \lesssim \lambda^{1+(2n-2)/M_\Omega} \sigma(B(\zeta_0, \lambda r)), \tag{35}$$

As we said before, all these definitions are local, and may be given in the context of H-domains.

As in the case of the unit ball, if w_j are the coordinates of $w - z$ in the basis $\{v^{(1)}, \ldots, v^{(n)}\}$ and if $w_j = s_j + it_j$, then s_j for $j \geq 2$ and t_j for $j \geq 1$ define $2n - 1$ local coordinates of $\partial\Omega$ in a neighborhood of z. We will still speak of *special coordinates at the point z and the scale r*.

In the neighborhood of $z \in \partial\Omega$, the hypersurface $\partial\Omega$ coincides with the graph $\Re w_1 = h(\Im w_1, w')$, with $w' = (w_2, \cdots, w_n)$. As in the case of the unit ball, we are interested in estimates on $D_w^{(\alpha,\beta)} S(\,(h(t_1, s' + it') + it_1, w'))$, where α is an $n-1$-index of derivation in the variable s', while β is an n-index of derivation in t. The equivalent of (26) is given by the estimates of McNeal and Stein [McS1]

and [McS2], see also [BPS1], Lemma 4.7 for an analogous context. For $d(w,z) < r$ and $\zeta \notin B(z, Cr)$, we have

$$|D_w^{(\alpha,\beta)} S(\zeta, (h(t_1, s'+it')+it_1, w'))| \lesssim \tau^{-(1+\beta_1, 2+\alpha_2+\beta_2, \cdots, 2+\alpha_n+\beta_n)}(z, d(w,z)). \tag{36}$$

As in [BPS1], we will also use the existence of a support function given by Diederich and Fornaess [DFo].

Theorem 6.2. *Let Ω be a smoothly bounded pseudoconvex H-domain of finite type in $\mathbb{C}^n$. Then there exist a neighborhood U of the boundary $\partial\Omega$ and a function $H \in \mathcal{C}^\infty(\mathbb{C}^n \times U)$ such that the following conditions hold:*

(i) $H(\cdot, w)$ is holomorphic on Ω for all $\zeta \in U$;
(ii) there exists a constant $c_1 > 1$ such that

$$\frac{1}{c_1} d(z,w) \le |H(z,w)| \le c_1 d(z,w).$$

With all these definitions, we claim the following.

Statement of results for H-domains. *The analogues of Theorems 1.3 to Corollary 1.20 are valid for the H-domain Ω with the following modifications: $N_p := (\frac{1}{p}-1)(M_\Omega+2n-2)-1$ in Definition 1.5; in Proposition 1.9, the condition is $L < \frac{2N+2}{M_\Omega}$, while in Proposition 1.10, we have $L_p := (2/p-2)\left(1+\frac{2n-2}{M_\Omega}\right)$. Finally, for the definition of BMO(ϱ), we have to take $N+1 > \alpha(M_\Omega+2n-2)$.*

Let us sketch the modifications to be done. Atoms adapted to a ball $B := B(\zeta_0, r_0)$ are defined as before, using special coordinates at ζ_0 and at scale r_0 to define the vanishing moment conditions. Remark that the coordinates depend on r_0, but the space $\mathcal{P}_N(\zeta_0)$ does not.

Then, in Lemma 3.1, the second estimate has to be replaced by

$$|A(\zeta)| \lesssim \left(\frac{r_0}{d(\zeta,\zeta_0)}\right)^{\frac{N+1}{M_\Omega}} \frac{\|a\|_2 \sigma(B)^{\frac{1}{2}}}{\sigma(\zeta_0, d(\zeta,\zeta_0))} \quad \text{for } d(\zeta,\zeta_0) \ge Cr_0. \tag{37}$$

The proof is the same, using the estimates (36) in place of (26).

Next, molecules are defined as follows.

Definition 6.3. *A holomorphic function $A \in \mathcal{H}^2(\Omega)$ is called a molecule of order L, associated to the ball $B := B(z_0, r_0) \subset \partial\Omega$, if it satisfies*

$$\sup_{\varepsilon<\varepsilon_0} \int_{\partial\Omega} \left(1 + \frac{d(z_0,\xi)^L}{r_0^L} \times \frac{\sigma(B(z_0, d(z_0,\xi)))}{\sigma(B(z_0,r_0))}\right) |A(\xi - \varepsilon\nu(\xi))|^2 \frac{d\sigma(\xi)}{\sigma(B)} < \infty, \tag{38}$$

with ν the outward normal vector. In this case, the left hand side is $\|A\|^2_{\mathrm{mol}(B,L)}$. It follows from (37), cutting the integral into dyadic balls, that the projection of an atom related to the ball $B := B(z_0, r_0) \subset \partial\Omega$ is a molecule of order $L < \frac{2N+2}{M_\Omega}$.

Finally, to see that a molecule of order L is in the Hardy space $\mathcal{H}^\Phi$, we prove that, with $B_k := B(z_0, 2^k r_0)$,

$$\int_{B_k\setminus B_{k-1}} \Phi(g)\frac{d\sigma}{\sigma(B)} \lesssim 2^{-k\varepsilon}\Phi\left(\left(2^{kL}\frac{\sigma(B_k)}{\sigma(B)}\int_{B_k\setminus B_{k-1}} g(\xi)^2\frac{d\sigma(\xi)}{\sigma(B)}\right)^{1/2}\right)$$

for some $\varepsilon > 0$. To prove this last inequality, we use again Jensen Inequality (24) for the measure $d\sigma$ on B_k, divided by its total mass $\sigma(B_k)$. This gives

$$\int_{B_k\setminus B_{k-1}} \Phi(g)\frac{d\sigma}{\sigma(B)} \lesssim \frac{\sigma(B_k)}{\sigma(B)}\Phi\left(\left(\int_{B_k\setminus B_{k-1}} g(\xi)^2\frac{d\sigma(\xi)}{\sigma(B_k)}\right)^{1/2}\right).$$

We conclude by using the fact that Φ is of lower type p, which allows to write that

$$\frac{\sigma(B_k)}{\sigma(B)}\Phi(t) \lesssim \Phi\left(\left(\frac{\sigma(B_k)}{\sigma(B)}\right)^{\frac{1}{p}} t\right).$$

Using (35), one finds that it is sufficient to choose $L > L_p := 2(1/p-1)\left(1 + \frac{2n-2}{M_\Omega}\right)$.

Up to now, we have given the modifications for having the atomic decomposition, the continuity of the Szegö projection, the duality. It remains to see the modifications in the proof of the factorization theorem. As at the beginning of Section 5, we factorize each molecule A associated to a ball $B := B(\zeta_0, r)$ as $B = fg$, with B a molecule and g a $BMOA$-function.

For this factorization, we use the support function given in Theorem 6.2. We set $H_0 = H(\cdot, \tilde\zeta_0)$, where $\tilde\zeta_0 = \zeta_0 - r\nu(\zeta_0)$. We choose $g = \log(cH_0^{-1})$ with c so that g does not vanish in Ω.

We have as before the inequality

$$\|f\|_{\mathrm{mol}(B,L')} \lesssim \frac{\|A\|_{\mathrm{mol}(B,L)}}{\log(e + \sigma(B)^{-1})} \tag{39}$$

for $L' < L$. Just use (ii) in Theorem 6.2.

We now prove that $\log(cH_0^{-1})$ belongs to $BMOA$ with bounds independent of ζ_0 and r. The proof follows the same line as the one in the unit ball, using that $|H_0|$ and $|\nabla H_0$ are uniformly bounded in Ω.

This finishes the proof of the factorization theorem.

□

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